

# A general Reynolds analogy theory for the compressible wall-bounded turbulence

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A general Reynolds analogy (GRA) theory is proposed for the mean and fluctuating velocity and temperature in compressible wall-bounded turbulent flows. In particular, an exact analogy solution is derived for compressible turbulent pipe and channel flows and an approximate analogy solution is derived for compressible turbulent boundary layers (CTBL), both of which are independent of fluid Prandtl number and wall temperature condition. The analogy solutions are in excellent agreement with direct numerical simulation data, able to reproduce empirical relations, and can be viewed as extensions of existing theories. In contrast to Walz's equation for adiabatic CTBL, the mean temperature-velocity relation derived by GRA can be applied to different wall-bounded flows in non-adiabatic wall condition, which is achieved by extending Walz's adiabatic recovery factor to a heat flux dependent one. The fluctuation temperature-velocity relations derived by GRA are slightly different from the modified strong Reynolds analogy derived phenomenologically by Huang et al. (HSRA), and have a better performance than HSRA. In addition, several key quantities are introduced in GRA, including a general total enthalpy (or temperature) and an adiabatic degree—a well-founded dimensionless parameter for characterizing the wall-temperature effects in non-adiabatic flows. The GRA unveils the universal feature behind the complex nonlinear couplings between the thermal and velocity fields, and makes possible of predicting the mean fields of compressible wall-bounded turbulence with the information of the corresponding incompressible flow.

**Key words:** Compressible turbulence, Shear layer turbulence, Turbulence theory, Compressible boundary layers

## 1. Introduction

There are strong nonlinear couplings between the velocity field and the temperature field of compressible wall-bounded turbulent flows (Gaviglio 1987; Smits & Dussauge 2006). The temperature-velocity relationship is crucial to the design of high speed vehicles; thus it has been the focus of many studies during the past decades (Walz 1969;

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Cebeci & Smith 1974; Gaviglio 1987; Huang *et al.* 1995; Duan & Martín 2011). Nevertheless, it remains an unresolved question owing to the complex character of compressible turbulence, and owing to the various wall temperature conditions (adiabatic/non-adiabatic walls) and flow situations (external/internal flows) involved in aerospace engineering.

The first theory on the temperature-velocity relationship was presented by O. Reynolds in 1874 for incompressible laminar boundary layers (Reynolds 1961). Then it was extended to incompressible turbulent boundary layers (ITBL). For a zero-pressure-gradient (ZPG) thermal ITBL, the extension gives a linear dependence of temperature on velocity, a result of the similarity between the momentum equation and the energy equation, i.e. the so-called Reynolds analogy (Gaviglio 1987). The similarity is largely attributed to the negligible viscous dissipation in incompressible turbulence (Gaviglio 1987). In compressible flow, however, the viscous dissipation plays an important role (Gaviglio 1987; Smits & Dussauge 2006), prohibiting a straightforward analogy.

A lot of studies were conducted to extend the Reynolds analogy to compressible flows. In terms of the relationship between mean temperature and mean velocity, several relations were derived based on various approximations. Busemann (1931) and Crocco (1932) obtained the first relation (Pirozzoli *et al.* 2004) for ZPG compressible laminar boundary layers by assuming unity Prandtl number (Morkovin 1962; Gatski 2009; compressibility (for air,  $Pr \approx 0.7$ ). Their derivation can be applied to turbulent flows. In contrast to the linear relation of Reynolds analogy, the Crocco-Busemann relation exhibits a quadratic dependence of mean temperature on mean velocity:

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_\delta} \frac{\bar{u}}{\bar{u}_\delta} + \frac{\bar{T}_\delta - \bar{T}_r}{\bar{T}_\delta} \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2, \quad \bar{T}_r = \bar{T}_\delta + \bar{r} \frac{\bar{u}_\delta^2}{2C_p}, \quad (1.1)$$

where  $u$  is streamwise velocity,  $T$  is temperature,  $C_p$  is specific heat at constant pressure,  $\bar{\cdot}$  denotes Reynolds average and  $\bar{r}$  is called extended recovery factor here to distinguish from the classical recovery factor  $r$  (Walz 1969). Throughout this paper, subscript  $w$  denotes wall and  $\delta$  denotes boundary layer thickness or channel/pipe center. In the Crocco-Busemann relation,  $\bar{r}$  is unity, which was modified to  $r \approx 0.9$  by Walz (1966) to account for the deviation of from unity (Walz 1969). Duan *et al.* (Duan & Martín 2011) recently presented an empirical relation based on a vast amount of direct numerical simulation (DNS) data up to hypersonic regime. Duan's relation is in excellent agreement with DNS data and is valid for various flows, such as flows with low or high enthalpy and with or without chemical reaction. The only difference between Walz's equation and Duan's relation is that the first  $\bar{u}/\bar{u}_\delta$  in Eq. (1.1) is replaced by a quadratic function of  $\bar{u}/\bar{u}_\delta$ . As to be demonstrated in this paper, Duan's relation can be derived from our new theory, called general Reynolds analogy (GRA). The GRA gives a general mean temperature-velocity relation with a variable  $\bar{r}$  that is only associated with the wall-temperature condition and can be expressed as a function of  $r$ ,  $Pr$ ,  $s$  and  $\Theta$ , where  $s$  is the well-known Reynolds analogy factor (Bradshaw 1977) and  $\Theta = (\bar{T}_w - \bar{T}_\delta)/(\bar{T}_r - \bar{T}_\delta)$  is a newly defined adiabatic degree based on the recovery temperature (i.e. the adiabatic wall temperature)  $\bar{T}_r = \bar{T}_\delta + r\bar{u}_\delta^2/(2C_p)$ . For adiabatic walls, the general mean temperature-velocity relation naturally reduces to Walz's equation (Walz 1969).

As to the fluctuation fields, a series of relations between the streamwise velocity fluctuation  $u'$  and the temperature fluctuation  $T'$  were also derived. The first relations were identified by Morkovin in 1962 (primarily due to Young in 1951 (Spina *et al.* 1994)) and were known collectively as Strong Reynolds Analogy (SRA). SRA were derived for ZPG, adiabatic CTBL under two assumptions: (a)  $Pr = 1$  (Guarini *et al.* 2000) or  $Pr_m = 1$

(Smits & Dussauge 2006), and (b)  $T'_t \approx 0$  (Morkovin 1962) or  $T'_t \propto u'$  (Guarini *et al.* 2000), where  $T_t = T + u_i u_i / (2C_p)$  is total temperature. In 1974, SRA was extended to non-adiabatic walls by Cebeci and Smith (Cebeci & Smith 1974) and was called extended SRA (ESRA). Owing to the too rough assumptions (Morkovin 1962; Gaviglio 1987; Spina *et al.* 1994; Guarini *et al.* 2000; Maeder *et al.* 2001; Smits & Dussauge 2006; Duan *et al.* 2010), both SRA and ESRA deviate clearly from experimental (Gaviglio 1987) and DNS data (Guarini *et al.* 2000; Maeder *et al.* 2001; Duan *et al.* 2010, 2011), hence are inappropriate for applying to turbulence models. To avoid these inappropriate assumptions, some modified SRA were proposed by several authors, including Gaviglio (GSRA, 1987), Rubesin (RSRA, 1990) and Huang (HSRA, 1995). These modified SRA have a common form:

$$\frac{\sqrt{T'^2}/\bar{T}}{(\gamma - 1) M^2 \sqrt{u'^2}/\bar{u}} = \frac{1}{c(1 - \partial \bar{T}_t / \partial \bar{T})}, \quad (1.2)$$

where  $c = 1$  for GSRA,  $c = 1.34$  for RSRA and  $c = Pr_t$  for HSRA,  $\gamma = C_p/C_v = 1.4$  is the ratio of specific heat,  $M = \bar{u}/\sqrt{\gamma R \bar{T}}$  is local Mach number. Among the three relations, HSRA best agrees with DNS data at various wall temperature conditions and for various flows including CTBL (Guarini *et al.* 2000; Maeder *et al.* 2001; Pirozzoli *et al.* 2004; Duan *et al.* 2010, 2011; Duan & Martín 2011) and compressible channel flow (CCF) (Huang *et al.* 1995). Guarini *et al.* (Guarini *et al.* 2000) later presented a thorough analysis and pointed out that HSRA revealed the key analogy between the normalized rates of turbulent heat and momentum transfer. More specifically, the correlation coefficients  $R_{v'u'}$  and  $R_{u'T'}$  are nearly equal throughout the boundary layer, where  $R_{\alpha'\beta'} = \sqrt{\alpha'\beta'}/(\sqrt{\alpha'^2}\sqrt{\beta'^2})$ . In this paper, we will prove that this key analogy is a consequence of GRA, and a modified HSRA (MHSRA) is resulted, which demonstrates a better performance than HSRA.

All the previous analogy theories hold only for ZPG flows. For flow driven by pressure gradient, such as channel and pipe flows, the streamwise pressure gradient excludes a straightforward application of the analogy theory. As to be shown, GRA presents as a unified theory for a resolution of the problem. In addition, even for ZPG CTBL, the existing theories have more or less defects. For example, both SRA and the modified SRA yield that  $R_{u'T'} = -1$  for CTBL and  $R_{u'T'} = 1$  for CCF and compressible pipe flow (CPF), which, however, disagree with experimental (Guarini *et al.* 2000; Smits & Dussauge 2006) and DNS data (Huang *et al.* 1995; Guarini *et al.* 2000; Maeder *et al.* 2001; Pirozzoli *et al.* 2004; Duan *et al.* 2010, 2011; Duan & Martín 2011). The disagreement has not been well understood, but can be explained by GRA.

In this paper, we present the GRA for compressible wall-bounded turbulent flows. An exact analogy solution for CCF/CPF and an approximate analogy solution for CTBL are obtained, which are consistent with the existing theories and the empirical relations, and in excellent agreement with DNS data. For the DNS data, a notation like *CCFM4.50* $\Theta$ 1.00 is used to denote a CCF with  $M = 4.50$  and  $\Theta = 1.00$ .  $x, y, z$  (or  $x_1, x_2, x_3$ ) are used to denote streamwise (axial), wall-normal (radical) and spanwise (azimuthal) directions in Cartesian (cylindrical) coordinate, and the corresponding velocity components are  $u, v, w$  (or  $u_1, u_2, u_3$ ). Throughout the paper, Reynolds decomposition is used only, but all the derivations and conclusions can be extended to Favre decomposition (Gatski & Bonnet 2009) in a straightforward way. This paper is structured as follows. A brief review and discussion of the existing Reynolds analogy theories are presented in section 2. The GRA is presented in section 3, followed by the mean temperature-velocity

relationship in sec.4 and the fluctuation temperature-velocity relationship in sec.5. A discussion of GRA is presented in sec.6. And finally, conclusions are drawn in sec.7.

## 2. A review of various Reynolds analogies

The time-averaged Navier-Stokes equations read (Gatski & Bonnet 2009)

$$\partial_{x_i}(\overline{\rho u_i}) = 0 \quad (2.1)$$

$$\partial_{x_j}(\overline{\rho u_i u_j}) = -\partial_{x_i} \bar{p} + \partial_{x_j}(\overline{\tau_{x_i x_j}}) \quad (2.2)$$

$$\partial_{x_j}(\overline{\rho h_t u_j}) = \partial_{x_j}(\overline{u_i \tau_{x_i x_j}}) - \partial_{x_j}(\overline{q_{x_j}}) \quad (2.3)$$

,where Eq. (2.1) is continue equation, Eq. (2.2) is momentum equation and Eq. (2.3) is energy equation,  $\rho$  is density,  $p$  is pressure,  $h_t = C_p T_t = h + u_i u_i / 2$  is total enthalpy and  $h = C_p T$  is enthalpy,  $\tau_{x_i x_j} = 2\mu(S_{ij} - S_{kk}\delta_{ij}/3)$  is viscous stress tensor with  $S_{ij} = (\partial_{x_j} u_i + \partial_{x_i} u_j)/2$ ,  $q_{x_i} = -k\partial_{x_i} T$  is heat flux,  $\mu$  is viscosity that is related with thermal conductivity  $k$  by  $Pr = C_p \mu / k$ . Throughout the paper, the following approximations are frequently adopted, which are highly accurate and well accepted:

$$u_i u_i \approx u^2; \overline{u u} \approx \bar{u}^2; u^2 - \bar{u}^2 \approx 2\bar{u}u'; \overline{(\rho v)'(u^2/2)} \approx \bar{u}(\overline{\rho v})'u'. \quad (2.4)$$

In the following, we briefly review the existing Reynolds analogy theories. To be concise, discussions are limited to ZPG turbulent boundary layers.

### 2.1. Reynolds analogy for incompressible turbulent boundary layers

For ZPG incompressible turbulent boundary layers, the momentum and energy equations can be simplified to (Gaviglio 1987)

$$\bar{\rho} \bar{u} \partial_x \bar{u} + \bar{\rho} \bar{v} \partial_y \bar{u} = \partial_y (\bar{\mu} \partial_y \bar{u} - \overline{\rho u' v'}) \quad (2.5)$$

$$\bar{\rho} \bar{u} \partial_x \bar{T} + \bar{\rho} \bar{v} \partial_y \bar{T} = \partial_y (\bar{\mu} / Pr \partial_y \bar{T}) + \partial_y (\overline{-\rho T' v'}) + \Phi / C_p \quad (2.6)$$

,where  $\Phi = 2\mu S_{ij} S_{ji}$  is viscous dissipation and negligible in incompressible flows (Cebeci & Bradshaw 1984). Above the viscous sublayer, turbulent transport of momentum and heat predominates over molecular transport, so the molecular transport terms can also be neglected. One then obtains:

$$\bar{\rho} \bar{u}^* \partial_x \bar{u}^* + \bar{\rho} \bar{v}^* \partial_y \bar{u}^* = -\partial_y \overline{\rho u^{*'} v^{*'}} \quad (2.7)$$

$$\bar{\rho} \bar{u}^* \partial_x \bar{T}^* + \bar{\rho} \bar{v}^* \partial_y \bar{T}^* = -\partial_y \overline{\rho T^{*'} v^{*'}} \quad (2.8)$$

,where  $u_i^* = u_i / u_\delta$  and  $T^* = (\bar{T} - T_w) / (T_\delta - T_w)$ . The corresponding boundary conditions are (a).  $y = 0 : u^* = T^* = 0$  (b).  $y = \delta : u^* = T^* = 1$ . The similarity between Eq.(2.8) and Eq.(2.7) means  $u^* = T^*$ . In the dimensional form, the temperature-velocity relations are

$$(\bar{T} - T_w) / (T_\delta - T_w) = \bar{u} / u_\delta \quad (2.9)$$

$$T' / (T_\delta - T_w) = u' / u_\delta. \quad (2.10)$$

Eq. (2.9) and Eq. (2.10) show a linear dependence of temperature on velocity for both the mean and fluctuation fields. As pointed out by Cebeci and Bradshaw (Cebeci & Bradshaw 1984), the instantaneous fluctuation relation of Eq. (2.10) denotes an 'exact analogy' that only applies to an ideal, i.e. unrealistic, situation. A weak form can be derived from Eq. (2.10):

$$\sqrt{\overline{T'^2}} / (T_\delta - T_w) \approx \sqrt{\overline{u'^2}} / u_\delta, \quad (2.11)$$

which is less questionable.

## 2.2. Crocco-Busemann relation (1931,1932), SRA (1962), and ESRA (1974)

In compressible flows, the viscous dissipation can not be neglected, so the 'Reynolds analogy' for incompressible flows cannot be applied directly. Owing to Young (Howarth 1953), the energy equation can be written in the form of total enthalpy (Gatski & Bonnet 2009):

$$\begin{aligned} \bar{\rho}\bar{u}\partial_x\bar{h}_t + \bar{\rho}\bar{v}\partial_y\bar{h}_t = & \partial_y[(\bar{\mu}/Pr)\partial_y\bar{h}_t - \overline{\rho h'_t v'}] \\ & + \partial_y[\bar{\mu}(1 - 1/Pr)\partial_y(\bar{u}_i\bar{u}_i/2 - \overline{u'_i u'_i}/2)]. \end{aligned} \quad (2.12)$$

Assuming  $Pr = 1$ , momentum equation and energy equation show a similarity:

$$\bar{\rho}\bar{u}\partial_x\bar{u} + \bar{\rho}\bar{v}\partial_y\bar{u} = \partial_y(\bar{\mu}\partial_y\bar{u} - \overline{\rho u' v'}) \quad (2.13)$$

$$\bar{\rho}\bar{u}\partial_x\bar{h}_t + \bar{\rho}\bar{v}\partial_y\bar{h}_t = \partial_y(\bar{\mu}\partial_y\bar{h}_t - \overline{\rho h'_t v'}). \quad (2.14)$$

To find an analogy solution of Eq.(2.13) and Eq.(2.14), a rather rigorous assumption is introduced:  $h'_t = U_0 u'$ , where is a proportional constant with dimension of velocity. Subtracting Eq.(2.13) from Eq.(2.14) gives

$$(\bar{\rho}\bar{u}\partial_x + \bar{\rho}\bar{v}\partial_y - \partial_y\bar{\mu}\partial_y)(\bar{h}_t - U_0\bar{u}) = 0. \quad (2.15)$$

A *possible* analogy solution of Eq.(2.15) is  $\bar{h}_t - U_0\bar{u} = \text{const}$  in the whole  $x - y$  plane. Here, the emphasis of 'possible' is because this solution is a sufficient but unnecessary condition of Eq.(2.15). Applying the boundary conditions, one finds  $U_0 = (\bar{h}_{t\delta} - \bar{h}_{tw})/\bar{u}_\delta = -Pr\bar{q}_{yw}/\bar{\tau}_w$ . Then the mean temperature-velocity relation can be derived as:

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_{t\delta} - \bar{T}_w}{\bar{T}_\delta} \frac{\bar{u}}{\bar{u}_\delta} + \frac{\bar{T}_\delta - \bar{T}_{t\delta}}{\bar{T}_\delta} \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2; \bar{T}_{t\delta} = \bar{T}_\delta + \frac{\bar{u}_\delta^2}{2C_p}. \quad (2.16)$$

Eq. (2.16) is called Crocco-Busemann relation since it is similar to the relation derived by Crocco (1932) and Busemann (1931) for compressible laminar boundary layers.

For an adiabatic wall,  $\bar{q}_{yw} = 0$  gives  $U_0 = 0$ . The analogy solutions become  $\bar{T}_t \equiv \bar{T}_w$  and  $\bar{T}'_t \equiv 0$ . The former is moderately confirmed by DNS data (Guarini *et al.* 2000; Pirozzoli *et al.* 2004), which show the deviation of the mean total temperature from a constant is less than about 7%. The latter leads to:

$$T'_t \approx C_p T' + \bar{u} u' \approx 0. \quad (2.17)$$

, where the approximations in Eq. (2.4) are used. This instantaneous relation has several statistical consequences:  $\overline{u' T'} \approx -(\bar{u}/C_p)\overline{u'^2}$ ,  $\sqrt{\overline{T'^2}} \approx (\bar{u}/C_p)\sqrt{\overline{u'^2}}$ , and  $\overline{\rho T' v'} \approx -(\bar{u}/C_p)\overline{\rho u' v'}$ , and further yields:

$$\frac{\sqrt{\overline{T'^2}}/\bar{T}}{(\gamma - 1) M^2 \sqrt{\overline{u'^2}}/\bar{u}} \approx 1 \quad (2.18)$$

$$R_{u' T'} = \frac{\overline{u' T'}}{\sqrt{\overline{u'^2}} \sqrt{\overline{T'^2}}} \approx -1 \quad (2.19)$$

$$Pr_t = \frac{\overline{\rho u' v'} (\partial \bar{T} / \partial y)}{\rho T' v' (\partial \bar{u} / \partial y)} \approx 1 \quad (2.20)$$

$$\frac{\sqrt{\overline{T'^2}}}{\bar{T}_w - \bar{T}_\delta} \approx 2 \frac{\bar{u}}{\bar{u}_\delta} \frac{\sqrt{\overline{u'^2}}}{\bar{u}_\delta}. \quad (2.21)$$

The above relations are derived by using the approximation  $\bar{T}_t \approx C_p \bar{T} + \bar{u}^2/2$  and it's the normal invariance of  $\bar{T}_t$ , saying  $\bar{T}_t \equiv \bar{T}_w \equiv \bar{T}_\delta$ . Eq. (2.17)– Eq. (2.21) were first identified by Morkovin in 1962 and collectively known as *strong Reynolds analogy*. Here the use of 'strong' may come from the assumption  $h'_t = U_0 u'$  (Guarini *et al.* 2000) (or negligible total temperature fluctuation (Morkovin 1962; Lele 1994)), which is too rigorous to meet in a real turbulence.

The assumption was later invalidated by DNS data (Guarini *et al.* 2000; Maeder *et al.* 2001; Duan *et al.* 2011), which show that the magnitude of  $\sqrt{\overline{T_t'^2}}$  is comparable to  $\sqrt{\overline{T'^2}}$  (see Eq.(2.17)). Consequently, the SRA relations are poorly satisfied. For example, the predictions  $R_{u'T'} = -1$  and  $Pr_t = 1$  clearly deviate from the observed values of  $-R_{u'T'} = 0.5 \sim 0.7$  and  $Pr_t = 0.7 \sim 0.9$  in adiabatic CTBL up to hypersonic regime (Guarini *et al.* 2000; Maeder *et al.* 2001; Pirozzoli *et al.* 2004; Duan *et al.* 2011). However, Eq. (2.17) is well satisfied for both experiments (Gaviglio 1987) and numerical simulations (Guarini *et al.* 2000). Then a question arises as to how Eq. (2.17) can be satisfied under an incorrect assumption. As to this, both Debieve (1976) and Gaviglio (1987) (Gaviglio 1987) demonstrated that Eq. (2.17) was a sufficient but unnecessary condition of Eq. (2.18). By rearranging the definition of the total temperature fluctuations as

$$\frac{\sqrt{\overline{T'^2} - \overline{T_t'^2} + 2\overline{T_t'T'}/\bar{T}}}{(\gamma - 1)M^2\sqrt{\overline{u'^2}/\bar{u}}} = 1 \quad (2.22)$$

, Guarini (2000) (Guarini *et al.* 2000) pointed out that the validity of Eq. (2.17) came from  $\overline{T'^2} \gg \overline{T_t'^2} - 2\overline{T_t'T'}$ , instead of  $\overline{T_t'^2} \approx 0$ . Gaviglio (Gaviglio 1987) further observed that the total temperature fluctuation could be arranged in a more general form as

$$\sqrt{\overline{T_t'^2}} = [\overline{T'^2} + \overline{u'^2}(\bar{u}/C_p)^2 + 2(\bar{u}/C_p)\sqrt{\overline{u'^2}}\sqrt{\overline{T'^2}}R_{u'T'}]^{1/2}. \quad (2.23)$$

Applying Eq. (2.18) to Eq. (2.23), one obtains  $R_{u'T'} = \overline{T_t'^2}/2\overline{T'^2} - 1$  (Gaviglio 1987; Guarini *et al.* 2000; Pirozzoli *et al.* 2004), which has a better performance than Eq. (2.19).

SRA can be extended to non-adiabatic wall flows. Evaluating  $\bar{h}_t - U_0\bar{u} = \bar{h}_w$  at  $\delta$ , one finds  $\bar{T}_{t\delta} - \bar{T}_w = \bar{u}_\delta(U_0/C_p)$ . Substituting the relation into the assumption  $C_p T'_t = U_0 u'$ , with the aid of the approximations in Eq.(2.4), one obtains the following relations:

$$\frac{T'/\bar{T}}{(\gamma - 1)M^2\overline{u'}/\bar{u}} \approx 1 - C_p \frac{\bar{T}_{t\delta} - \bar{T}_w}{\bar{u}_\delta\bar{u}}. \quad (2.24)$$

$$\frac{\sqrt{\overline{T'^2}}/\bar{T}}{(\gamma - 1)M^2\sqrt{\overline{u'^2}}/\bar{u}} \approx 1 - C_p \frac{\bar{T}_{t\delta} - \bar{T}_w}{\bar{u}_\delta\bar{u}}. \quad (2.25)$$

The instantaneous relationship of Eq. (2.24) was proposed by Cebeci and Smith (Cebeci & Smith 1974) in 1974. The r.m.s. form of Eq.(2.24), i.e. Eq.(2.25), was suggested and referred to as extended SRA (ESRA) by Gaviglio (Gaviglio 1987) in 1987. In case of having heat flux at the wall, ESRA has a noteworthy improvement comparing with SRA, but has also an observable deviation from real turbulence (Gaviglio 1987).

### 2.3. Walz's equation (1966)

The deviation of  $Pr$  from unity is the major reason responsible for the difference between Crocco-Busemann relation and DNS data, as shown in fig.(2). To improve, Walz presented an approximate solution of the Reynolds averaged Navier-Stokes (RANS) equations with the assumption of constant mixing Prandtl number, i.e.  $Pr_m = const$ , which

is a reasonable approximation for CTBL and other shear flows (Smits & Dussauge 2006).  $Pr_m$  is defined by  $Pr_m = C_p(\bar{\mu} + \bar{\mu}_t)/(\bar{k} + \bar{k}_t)$ , where  $\bar{\mu}_t = (-\overline{\rho u'v'})/(\partial\bar{u}/\partial y)$  is eddy viscosity and  $\bar{k}_t = (-C_p\overline{\rho T''})/(\partial\bar{T}/\partial y)$  is eddy thermal conductivity.

In Walz's derivation, temperature is assumed a function of  $u$ , and RANS equations are written in terms of the independent variables  $x$  and  $u$ , instead of  $x$  and  $y$ . After neglecting most streamwise derivatives, the energy equation reads (Walz 1969; Smits & Dussauge 2006)

$$\bar{\tau}[\partial_{\bar{u}}(\partial_{\bar{u}}\bar{T}/Pr_m) + 1/C_p] + (1/Pr_m - 1)\partial_{\bar{u}}\bar{T}\partial_{\bar{u}}\bar{\tau} = 0 \quad (2.26)$$

, where  $\bar{\tau}$  is total shear stress. The boundary conditions are (a).  $\bar{u} = 0 : \bar{T} = \bar{T}_w, (\partial\bar{\tau}/\partial y)_{y=0} = dp/dx = 0$  (b).  $\bar{u} = \bar{u}_\delta : \bar{T} = \bar{T}_\delta, \bar{\tau} = \bar{\tau}_\delta = 0$ . The assumption  $Pr_m = \text{const}$  makes Eq. (2.26) can be integrated two times (Walz 1969; Smits & Dussauge 2006). Then one is able to obtain:

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_\delta} f_1 + \frac{\bar{T}_\delta - \bar{T}_r}{\bar{T}_\delta} f_2, \bar{T}_r = \bar{T}_\delta + r \frac{\bar{u}_\delta^2}{2C_p}, \quad (2.27)$$

where  $f_1$ ,  $f_2$  and  $r$  are functions of  $\bar{\tau}/\bar{\tau}_w$ ,  $Pr_m$  and  $\bar{u}/\bar{u}_\delta$  (Walz 1969). Using a linear approximation  $\bar{\tau}/\bar{\tau}_w \approx 1 - y/\delta$  and  $Pr_m \approx 0.86$  averaged over the boundary layer thickness, Walz (Walz 1969) found that  $f_1 \approx \bar{u}/\bar{u}_\delta$ ,  $f_2 \approx (\bar{u}/\bar{u}_\delta)^2$  and  $r \approx 0.9$  (0.88 in reference (Walz 1969)). Then Eq. (2.27) has the same form as that of Eq. (1.1), and was called modified Crocco-Busemann relation or Walz's equation.

#### 2.4. GSRA(1987), RSRA(1990), HSRA(1995)

To avoid the inappropriate assumptions made in SRA, Gaviglio attempted to establish a relationship between  $u'$  and  $T'$  based on the recognition of the universal and dominant role of the large scale motion in compressible wall-bounded turbulence. (Gaviglio 1987). He found that the fluctuating velocity and temperature induced by the large scale movements are proportional to the local gradient of mean velocity and temperature, and the corresponding proportional constant are assumed as velocity mixing length  $\ell_u = \sqrt{u'^2}/(\partial\bar{u}/\partial y)$  and temperature mixing length  $\ell_T = \sqrt{T'^2}/(\partial\bar{T}/\partial y)$ , saying  $u' = \ell_u(\partial\bar{u}/\partial y)$  and  $T' = \ell_T(\partial\bar{T}/\partial y)$ . In instantaneous and r.m.s. forms, one obtains  $cT'/\partial_y\bar{T} = u'/\partial_y\bar{u}$  and  $c\sqrt{T'^2}/\partial_y\bar{T} = \sqrt{u'^2}/\partial_y\bar{u}$ , where  $c = \ell_u/\ell_T$ . The latter relation, together with the approximations in Eq. (2.4), yields the modified SRA, i.e. Eq. (1.2). Based on experimental data (Gaviglio 1987), Gaviglio further assumed  $c = 1$ , which gave the GSRA. In 1990, Rubesin independently gave an equivalent form of GSRA with  $c = 1.34$ , a constant calculated by Huang et.al in 1995 (Huang *et al.* 1995) using various turbulence modeling constants given by Rubesin (see Eq.(20) in reference (Rubesin 1990)). In Contrast to the constant value of  $c$ , Huang et.al (Huang *et al.* 1995) (1995) showed that  $c$  is equal to  $Pr_t$  that varies along the wall-normal direction. Indeed, multiplying both sides of  $cT'/\partial_y\bar{T} = u'/\partial_y\bar{u}$  by  $\rho v'$  and averaging, one obtains  $c = (\overline{\rho v' u' \partial_y \bar{T}})/(\overline{\rho v' T' \partial_y \bar{u}}) = Pr_t$ , which gives HSRA.

Possibly owing to the universality of the large scale movements in wall-bounded turbulence, HSRA works well for various wall-bounded turbulent flows at various wall conditions. In channel flow with cold wall, Huang et.al (Huang *et al.* 1995) showed HSRA had an evident improvement over ESRA. In ZPG CTBL, Duan's DNS data up to  $M = 12$  showed that HSRA was effective for adiabatic and non-adiabatic, catalytic and non-reacting walls at low- and high- enthalpy conditions (Duan *et al.* 2010, 2011; Duan & Martín 2011).

2.5. *The empirical temperature-velocity relation (2011)*

Recently, Duan et.al (Duan & Martín 2011) successfully removed the explicit dependence of the temperature-velocity relationship on thermal and chemical models by an empirical relation, in which a dimensionless 'recovery enthalpy' is introduced. For a calorically perfect gas,  $\bar{h}_t = C_p \bar{T}$ , the recovery enthalpy degenerates to a dimensionless recovery temperature  $\bar{T}_{ry}^*$  defined as  $\bar{T}_{ry}^* = (\bar{T}_{ry} - \bar{T}_w)/(\bar{T}_\delta - \bar{T}_w)$  with  $\bar{T}_{ry} = \bar{T} + r\bar{u}^2/(2C_p)$ . Plotting  $\bar{T}_{ry}^*$  versus  $\bar{u}/\bar{u}_\delta$ , Duan et.al found that all DNS data collapsed together, independent on freestream Mach number, wall temperature, surface catalysis and enthalpy conditions. The best fitting of  $\bar{T}_{ry}^* = f(\bar{u}/\bar{u}_\delta)$  gives

$$f\left(\frac{\bar{u}}{\bar{u}_\delta}\right) = (1 - \alpha)\left(\frac{\bar{u}}{\bar{u}_\delta}\right)^2 + \alpha\left(\frac{\bar{u}}{\bar{u}_\delta}\right), \alpha = 0.8259. \quad (2.28)$$

By expanding  $\bar{T}_{ry}^* = f(\bar{u}/\bar{u}_\delta)$ , Duan et.al (Duan & Martín 2011) obtained a mean velocity -temperature relation in the following form:

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_\delta} f\left(\frac{\bar{u}}{\bar{u}_\delta}\right) + \frac{\bar{T}_\delta - \bar{T}_r}{\bar{T}_\delta} \left(\frac{\bar{u}}{\bar{u}_\delta}\right)^2, \bar{T}_r = \bar{T}_\delta + r\frac{\bar{u}_\delta^2}{2C_p}. \quad (2.29)$$

Comparing Eq. (2.29) with Walz's equation Eq. (2.27), the only difference is the replacement of  $\bar{u}/\bar{u}_\delta$  in the second term of Walz's relation with  $f(\bar{u}/\bar{u}_\delta)$ .

### 3. General Reynolds analogy for compressible wall-bounded turbulence

Before proceeding, we recall the key approximations and assumptions involved in the previous analogy theories, which result in the final differences between the theories and real turbulence. In Crocco-Busemann relation, SRA, and ESRA, an approximation of  $Pr = 1$  and an assumption of  $H' = U_0 u'$  are introduced. Both of them are not good descriptions of real turbulence. Young (1986) pointed out that the difference caused by  $Pr - 1 \neq 0$  represents a measurement of the dissimilarity of the two different modes of transfer for vector  $\rho \vec{u}$  and scalar  $T_t$  which do not respond in the same manner to change in density and pressure (Gaviglio 1987). The assumption  $H' = U_0 u'$  is also an 'exact' analogy similar to Reynolds analogy. A consequence of this 'exact' equation of fluctuations, as well as the fluctuation assumptions to derive GSRA and HSRA, is the rigorous relation of  $|R_{u'T'}| = 1$ , which is unfortunately not satisfied in a real wall-bounded turbulence. As for Walz's equation, a lot of DNS data show that it is satisfactory in quasi-adiabatic CTBL (Pirozzoli *et al.* 2004; Duan *et al.* 2011), but unsatisfactory (Duan *et al.* 2010; Duan & Martín 2011) in non-adiabatic flows, as shown in Fig.(2). Two key approximations are made to derive Walz's equation: (a)  $Pr_m = const$ , (b)  $\bar{\tau}/\bar{\tau}_w \approx 1 - y/\delta$ . Since  $f_1$ ,  $f_2$  and  $r$  are complex functions of  $Pr_m$  and  $\bar{\tau}/\bar{\tau}_w$ , it is difficult to quantify the error introduced by these two assumptions, especially for (b). However, our DNS data show that  $1 - \bar{\tau}/\bar{\tau}_w \approx (y/\delta)^n$  for quasi-adiabatic wall, where  $n \approx 1.35$  at  $M = 2.25$  and  $n \approx 1.10$  at  $M = 6.00$ . For an over cooled wall, a near-wall overshoot of the total stress ( $\bar{\tau}/\bar{\tau}_w > 1$ ) is observed, also inconsistent with the linear approximation of (b).

All in all, an improved theory should avoid, if possible, those inappropriate approximations or assumptions. Besides, the theory should, ideally, yield the empirical relations of Duan et al. and Huang et al. In this section, we present a general analogy between the Reynolds-averaged momentum and energy equations that meets the above requirements.



### 3.1. Compressible pipe flow

For a compressible pipe flow, the governing equations of a thermally and calorically perfect gas in cylindrical coordinates are referred to reference (Ghosh *et al.* 2010). Conventionally, the radial coordinate and velocity are denoted by  $r$  and  $u_r$ , respectively, the former of which is used already by recovery factor in this paper. To avoid ambiguity, we also use  $y$  and  $v$  to denote them. This is helpful to demonstrate the similarities and differences between pipe and channel flows. Note that some  $y$  in this subsection have been marked as  $\hat{y}$  (i.e.  $\hat{y} = y$ ), which is used to simplify the derivation of GRA in channel flow (see subsection. 3.2). Then the momentum and energy equations averaged in time and in axial and azimuthal directions are written as (see Appendix A):

$$\hat{y}\partial_x\bar{p} = \partial_y[\hat{y}(\overline{\tau_{xy}} - \overline{(\rho v)'u'})] \quad (3.1)$$

$$0 = \partial_y[\hat{y}(\overline{u\tau_{xy}} - \overline{(\rho v)'h'_t} + \overline{v\tau_{yy}} + \overline{k\partial_y T})]. \quad (3.2)$$

Using approximations of  $\overline{u\tau_{xy}} \approx \bar{u}\bar{\tau}_{xy}$ ,  $\bar{\tau}_{xy} \approx \bar{\mu}\partial_y\bar{u}$ ,  $\bar{q}_y = -\bar{k}\partial_y\bar{T} \approx -\bar{k}\partial_y\bar{T}$  and approximations in Eq. (2.4), neglecting the small term  $\partial_y(\overline{v\tau_{yy}})$  and defining total stress as  $\bar{\tau} = \bar{\mu}\partial_y\bar{u} - \overline{(\rho v)'u'}$ , Eq. (3.1) and Eq. (3.2) can be simplified to

$$\hat{y}\partial_x\bar{p} \approx \partial_y[\hat{y}\bar{\tau}] = \partial_y[\hat{y}(\bar{\mu}\partial_y\bar{u} - \overline{(\rho v)'u'})] \quad (3.3)$$

$$0 \approx \partial_y[\hat{y}(\bar{u}\bar{\mu}\partial_y\bar{u} - \overline{(\rho v)'h'_t} + \overline{k\partial_y T})]. \quad (3.4)$$

To search for a general analogy solution between the momentum and energy equations, an extended total enthalpy/temperature, denoted as  $h_e/T_e$ , is defined here to link velocity and temperature fields

$$h_e = C_p T_e = C_p T + \bar{r}(y) \frac{u^2}{2}, \quad (3.5)$$

where  $\bar{r}(y)$  is called extended recovery function and to be determined later. Using this definition and applying the approximations in Eq. (2.4), Eq. (3.4) can be rearranged to (see Appendix B)

$$\partial_y[\hat{y}((\bar{r} - 1)\bar{u}\bar{\tau} + (\bar{\mu}\bar{u}^2/2)\partial_y\bar{r} + (1 - \text{Pr})\bar{q}_y)] \approx \partial_y[\hat{y}(\bar{\mu}\partial_y\bar{h}_e - \overline{(\rho v)'h'_e})]. \quad (3.6)$$

, or integrated from center to to give  $\hat{y}(\bar{u}\bar{\tau} - C_p\overline{(\rho v)'T'} - \bar{q}_y) = 0$ , where 0 comes from the symmetry in the pipe center. This means

$$\bar{u}\bar{\tau} = C_p\overline{(\rho v)'T'} + \bar{q}_y. \quad (3.7)$$

Similar to the derivation of SRA, a full analogy between the momentum equation Eq.(3.3) and the energy equation Eq.(3.6) can be guaranteed under the constraint of

$$\partial_y[\hat{y}((\bar{r} - 1)\bar{u}\bar{\tau} + (\bar{\mu}\bar{u}^2/2)\partial_y\bar{r} + (1 - \text{Pr})\bar{q}_y)] = U_w\hat{y}\partial_x\bar{p} \approx U_w\partial_y[\hat{y}\bar{\tau}], \quad (3.8)$$

where  $U_w$  is a proportional constant with dimension of velocity, and is only associated with wall conditions as demonstrated later. To avoid a rigorous proportional assumption on fluctuations as that in deriving SRA, we express  $h'_e$  in terms of  $u'$  by  $h'_e = U_w u' + \varepsilon'$ , where  $\varepsilon'$  characterizes the instantaneous deviation of real turbulence from the proportional relation  $h'_e = U_w u'$ . Then the energy equation Eq.(3.6) becomes

$$U_w\partial_y[\hat{y}\bar{\tau}] \approx \partial_y[\hat{y}(\bar{\mu}\partial_y\bar{h}_e - \overline{(\rho v)'h'_e} - \overline{(\rho v)'\varepsilon'})]. \quad (3.9)$$

Now we introduced a critical, also unique, assumption: comparing with  $h'_e$ , the deviation  $\varepsilon'$  is a high-order modification, and hence  $-\overline{(\rho v)'\varepsilon'}$  is negligible. Under this assumption, the momentum equation Eq. (3.3) and the energy equation Eq. (3.9) can be written

in a similar form as

$$\partial_y[\hat{y}(U_w\bar{\tau})] \approx \partial_y[\hat{y}(\bar{\mu}\partial_y(U_w\bar{u}) - U_w\overline{(\rho v)'}u')] \quad (3.10)$$

$$\partial_y[\hat{y}(U_w\bar{\tau})] \approx \partial_y[\hat{y}(\bar{\mu}\partial_y\bar{h}_e - U_w\overline{(\rho v)'}u')] \quad (3.11)$$

Subtracting Eq. (3.10) from Eq. (3.11) and integrating from wall to  $y$ , one has  $\hat{y}[\bar{\mu}\partial_y(\bar{h}_e - U_w\bar{\tau})] = \text{const}$ , where *const* equals zero because of the symmetry at the pipe center. A further integration along  $y$  gives

$$\bar{h}_e - U_w\bar{u} = \bar{h}_w. \quad (3.12)$$

It is worth pointing out the Eq.(3.12) is a *deterministic* solution because of the symmetry in pipe, in contrast to the *possible* solution discussed in SRA for CTBL where no symmetry exists. Combining the assumption  $h'_e = U_w u' + \varepsilon'$  and Eq.(3.12), we obtain a full analogy solution between the momentum equation and the energy equation:

$$h_e - U_w u - U_w \varepsilon' = \bar{h}_w. \quad (3.13)$$

Averaging Eq. (3.13) and then subtracting Eq. (3.12), we have  $\overline{\varepsilon'} = 0$ , which explains the reason for using  $'$  as the superscript in  $\varepsilon'$ .

So far, the extended recovery function  $\bar{r}(y)$  is undetermined. Expanding Eq. (3.12) with Eq. (3.5) and calculating its derivative with respect to  $y$ , one has:

$$C_p \partial_y \bar{T} + \bar{r} \bar{u} \partial_y \bar{u} + (\bar{u}^2/2) \partial_y \bar{r} = U_w \partial_y \bar{u}. \quad (3.14)$$

Applying the equation at the wall yields

$$U_w = C_p \left. \frac{\partial \bar{T}}{\partial \bar{u}} \right|_w = -\text{Pr} \frac{\bar{q}_{yw}}{\bar{\tau}_w}. \quad (3.15)$$

Substituting Eq. (3.14), Eq. (3.7) and Eq. (3.15) to Eq. (3.8),  $\bar{r}_y$  can be solved as (see Appendix C)

$$\bar{r}(y) = \frac{U_w}{\bar{u}} - \frac{C_p \overline{(\rho v)'} T'}}{\bar{u} \overline{(\rho v)'} u'}}, \quad (3.16)$$

and a *full* analogy solution is established.

### 3.2. Compressible channel flow

For compressible channel flow, Eq. (2.2) and Eq. (2.3) can be written as

$$\partial_x \bar{p} = \partial_y [\overline{\tau_{xy}} - \overline{(\rho v)'} u'] \quad (3.17)$$

$$0 = \partial_y [\overline{u \tau_{xy}} - \overline{(\rho v)'} h'_t + \overline{v \tau_{yy}} + \overline{k \partial_y T}]. \quad (3.18)$$

Notice that these two equations can be viewed as a consequence of setting  $\hat{y} = 1$  in Eq.(3.1) and Eq.(3.2); hence all the derivations and conclusions for pipe flow can be applied to channel flow by simply replacing  $\hat{y}$  with 1.

### 3.3. Compressible turbulent boundary layers at zero-pressure gradient

For a ZPG CTBL, Eq. (2.2) and Eq. (2.2) can be written as

$$\overline{p u} \partial_x \bar{u} + \overline{p v} \partial_y \bar{u} = \partial_y [\overline{\tau_{xy}} - \overline{(\rho v)'} u'] + \partial_x [\overline{\tau_{xx}} - \overline{(\rho u)'} u'] \quad (3.19)$$

$$\begin{aligned} \overline{p u} \partial_x \bar{h}_t + \overline{p v} \partial_y \bar{h}_t = & \partial_x (\overline{u \tau_{xx}}) + \partial_x (\overline{v \tau_{xy}}) + \partial_y (\overline{u \tau_{xy}}) + \partial_y (\overline{v \tau_{yy}}) \\ & - \partial_x \overline{(\rho u)'} h'_t - \partial_y \overline{(\rho v)'} h'_t + \partial_x (\overline{k \partial_x T}) + \partial_y (\overline{k \partial_y T}). \end{aligned} \quad (3.20)$$

Besides those approximates involved in pipe/channel flow, we also neglect the small terms associated with streamwise gradient:  $\partial_x(\overline{\tau_{xx}})$ ,  $\partial_x(\overline{v\tau_{xy}})$ ,  $\partial_x(\overline{(\rho u)'u'})$ ,  $\partial_x(\overline{u\tau_{xx}})$ ,  $\partial_x(\overline{(\rho u)'h'_t})$ ,  $\partial_x(\overline{k\partial_x T})$ . Then Eq. (3.19) and Eq. (3.20) can be simplified to

$$\overline{\rho u}\partial_x\bar{u} + \overline{\rho v}\partial_y\bar{u} \approx \partial_y[\bar{\mu}\partial_y\bar{u} - \overline{(\rho v)'u'}] = \partial_y\bar{\tau} \quad (3.21)$$

$$\overline{\rho u}\partial_x\bar{h}_t + \overline{\rho v}\partial_y\bar{h}_t \approx \partial_y[\bar{u}\bar{\mu}\partial_y\bar{h}_t - \overline{(\rho v)'h'_t}] + \partial_y(\bar{k}\partial_y\bar{T}). \quad (3.22)$$

In CTBL,  $\bar{\tau}$  can also be treated approximately as a function of only  $y$ , and the definitions for  $h_e$  and  $T_e$ , i.e. Eq. (3.5), still work. Using the approximations in Eq. (2.4), the energy equation Eq. (3.22) can be rewritten as

$$\overline{\rho u}\partial_x\bar{h}_e + \overline{\rho v}\partial_y\bar{h}_e \approx \partial_y(\bar{u}\bar{\mu}\partial_y\bar{h}_e - \overline{(\rho v)'h'_e}) + f_{res} \quad (3.23)$$

,where (see Appendix D)

$$f_{res} = \overline{\rho v}u^2\partial_y\bar{\tau}/2 + (\bar{\tau} - 1)\bar{u}\partial_y\bar{\tau} + \partial_y[(\text{Pr} - 1)\bar{q}_y - \bar{u}^2\bar{\mu}\partial_y\bar{\tau}/2 + (1 - \bar{\tau})\bar{u}\bar{\tau}]. \quad (3.24)$$

Similar to pipe/channel flows, a fully analogy between the momentum equation Eq. (3.21) and the energy equation Eq. (3.23) of ZPG CTBL requires  $f_{res} = 0$ , which can be satisfied by solving  $\bar{\tau}(y)$ . However, the exact solution of  $\bar{\tau}(y)$  is too complex to be determined analytically.

Here we look for an approximate solution of  $\bar{\tau}(y)$ . Using Eq. (3.14),  $f_{res} = 0$  can be rearranged and decomposed into  $f_{res} = f_{res1} + f_{res2} = 0$  (see Appendix E), where

$$f_{res1} = \partial_y[C_p(\overline{(\rho v)'T'}) - (U_w - \bar{\tau}\bar{u})\overline{(\rho v)'u'}] \quad (3.25)$$

$$f_{res2} = \partial_y[\bar{u}\bar{\tau} - C_p(\overline{(\rho v)'T'}) - \bar{q}_y] + [(\bar{\tau} - 1)\bar{u} - U_w]\partial_y\bar{\tau} + (\overline{\rho v}/\bar{\mu})[(U_w - \bar{\tau}\bar{u})\bar{\mu}\partial_y\bar{u} + \text{Pr}\bar{q}_y]. \quad (3.26)$$

Now we introduce a second assumption: *comparing with the other terms in Eq. (3.23),  $f_{res2}$  is negligible*. The validity of this assumption is discussed in section 4. Under this assumption, the constraint  $f_{res} = 0$  degenerates to  $f_{res1} = 0$  which can be integrated from wall to  $y$  to give

$$C_p(\overline{(\rho v)'T'}) - (U_w - \bar{\tau}\bar{u})\overline{(\rho v)'u'} = 0. \quad (3.27)$$

This relation is equivalent to the extended recovery function of Eq. (3.16). So, naturally, all the conclusions for pipe and channel flows also work in ZPG CTBL.

### 3.4. Adiabatic CTBL with pressure gradient

For CTBL with pressure gradient, the energy equation takes the same form as that of ZPG CTBL. For adiabatic wall, Eq. (3.15) gives  $U_w = 0$ . Provided that  $f_{res2}$  and  $\partial_x\bar{T}_w$  are still negligible in this flow,  $\bar{h}_e \equiv \bar{h}_w$  satisfies Eq. (3.23) independent on the pressure gradient (Gaviglio 1987). Therefore, the approximate analogy solution for ZPG flow is also suitable for adiabatic CTBL with pressure gradient.

### 3.5. General Reynolds analogy

So far, we obtain full analogy solutions for compressible pipe and channel flows, and approximate analogy solutions for ZPG CTBL under arbitrary wall-temperature conditions and for adiabatic CTBL with pressure gradient. These solutions have a common form of Eq. (3.13). To be simple and invariant, we define a general total enthalpy (temperature), denoted by  $h_g(T_g)$ , as

$$h_g = C_p T_g = C_p T + \bar{\tau}(y) \frac{u^2}{2} - U_w u; \bar{\tau}(y) = \frac{U_w}{\bar{u}} - \frac{C_p(\overline{(\rho v)'T'})}{\bar{u}(\overline{(\rho v)'u'})} \quad (3.28)$$

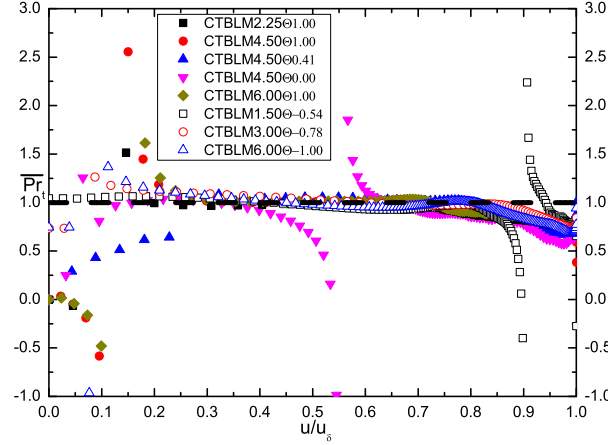


FIGURE 1. (color online). Variation of the modified turbulent Prandtl number for compressible turbulent boundary layer and compressible channel flow. The DNS of ZPG CTBLs and CCFs are simulated by us and Y.Z.Wang (private communication), respectively. For numerical detail, see reference Ref.(Li *et al.* 2001) and Ref.(Xin-Liang *et al.* 2006).

, where  $U_w$  takes the value of Eq.(3.15). Then the analogy solutions for compressible wall-bounded turbulence can be re-expressed with  $T_g$  as

$$C_p T_g - \varepsilon' \equiv C_p \bar{T}_w \quad (3.29)$$

$$\bar{T}_g \equiv \bar{T}_w \quad (3.30)$$

$$C_p T_g' = \varepsilon' \quad (3.31)$$

$$C_p \sqrt{T_g'^2} = \sqrt{\varepsilon'^2} \approx 0 \quad (3.32)$$

, where Eq. (3.32) is the statistical result of Eq. (3.31), and 0 is because  $\varepsilon'$  is a high-order perturbation on  $T_e'$  and  $u'$ . In this paper, Eq. (3.29)— Eq. (3.32) are collectively called general Reynolds analogies.

#### 4. Mean relationships based on the GRA

In this section, we discuss the relationship between mean velocity and mean temperature based on GRA. For convenience's sake, we first define a modified turbulent Prandtl number as

$$\bar{Pr}_t = \frac{\overline{(\rho v)' u'} (\partial \bar{T} / \partial y)}{\overline{(\rho v)' T'} (\partial \bar{u} / \partial y)} \approx Pr_t. \quad (4.1)$$

Using this definition and the approximations in Eq. (2.4), the mean relationship Eq. (3.30) can be expanded to give

$$\bar{T} - \frac{\bar{u}}{2} \left( \frac{1}{\bar{Pr}_t} \frac{\partial \bar{T}}{\partial \bar{u}} + \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) = \bar{T}_w, \quad (4.2)$$

which can be integrated in case of constant . For compressible wall-bounded turbulence (Huang *et al.* 1995; Ghosh *et al.* 2010; Duan *et al.* 2010, 2011; Duan & Martín 2011),  $\bar{Pr}_t$  is around one in the wall-normal direction, as shown in Fig. 1, where  $\bar{Pr}_t$  is plotted versus  $u/u_s$  in contrast to the traditional plot of  $Pr_t$  versus  $y/\delta$ . In the traditional plot, a wide plateau of  $\bar{Pr}_t \approx Pr_t \approx 0.8$  is observed in the outer region of a boundary layer, but

corresponding only to a small change of mean velocity. We argue that it is better to plot  $\overline{Pr}_t$  versus  $u/u_\delta$  because  $u/u_\delta$ , instead of  $y/\delta$ , is the variable in the temperature-velocity relationship, see Eq.(1.1). Only by this plot can the dependence of  $\overline{Pr}_t$  on the velocity variation be better revealed. As shown in Fig. 1,  $\overline{Pr}_t$  is around one up to  $u/u_\delta \approx 0.85$ . Hence, an approximation of  $\overline{Pr}_t = \text{const} = 1$  is used to integrate Eq. (4.2) from wall to to give

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \left( \frac{\bar{u}_\delta}{\bar{T}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) \left( \frac{\bar{u}}{\bar{u}_\delta} \right) + \left( \frac{\bar{T}_\delta - \bar{T}_w}{\bar{T}_\delta} - \frac{\bar{u}_\delta}{\bar{T}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2. \quad (4.3)$$

The derivative of Eq. (4.3) with respect to  $\bar{u}$  gives

$$\frac{\partial \bar{T}}{\partial \bar{u}} = \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \left( \frac{\bar{T}_w - \bar{T}_\delta}{\bar{u}_\delta^2/2} + \frac{2}{\bar{u}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) \bar{u}. \quad (4.4)$$

Applying Eq. (4.4) to Eq. (3.28) with the approximation  $\overline{Pr}_t = 1$ , one obtains

$$\begin{aligned} \bar{r}(y) &= \frac{U_w}{\bar{u}} - \frac{C_p \overline{(\rho v)' T'}}{\bar{u} \overline{(\rho v)' u'}} = \frac{C_p}{\bar{u}} \left( \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \frac{1}{\overline{Pr}_t} \frac{\partial \bar{T}}{\partial \bar{u}} \right) \\ &\approx \frac{C_p}{\bar{u}} \left( \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \frac{\partial \bar{T}}{\partial \bar{u}} \right) = \left( \frac{\bar{T}_w - \bar{T}_\delta}{\bar{u}_\delta^2/(2C_p)} + \frac{2C_p}{\bar{u}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right). \end{aligned} \quad (4.5)$$

Therefore, under the approximation  $\overline{Pr}_t = \text{const} = 1$ , the extended recovery function  $\bar{r}(y)$  becomes a constant and this constant only associated with the wall temperature condition. Furthermore, Eq. (4.5) reveals that  $\bar{r}(y)$  consists of two parts:  $\frac{\bar{T}_w - \bar{T}_\delta}{\bar{u}_\delta^2/(2C_p)}$  and  $\frac{2C_p}{\bar{u}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w$ . Similar to the recovery factor  $r$  defined for an adiabatic wall, the first part can be understood as a nominal recovery factor at a non-adiabatic isothermal wall. The second part denotes the contribution to  $\bar{r}(y)$  from the non-zero heat flux at the wall, which becomes important if the wall is over-heated or over-cooled, such as that in CCF or CPF. For an adiabatic wall,  $\bar{r}$  naturally reduces to the classical recovery coefficient, saying  $\bar{r} = r = (\bar{T}_r - \bar{T}_\delta)/(\bar{u}_\delta^2/2C_p)$ . Owing to these reasons,  $\bar{r}$  is called extended recovery factor, instead of function, hereinafter. Correspondingly, an extended recovery temperature can be defined as  $\bar{T}_{\bar{r}} = \bar{T}_\delta + \bar{r} \bar{u}_\delta^2/(2C_p)$ . With this definition, Eq.(4.3) can be rewritten as Equation Section (Next)

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_{\bar{r}} - \bar{T}_w}{\bar{T}_\delta} \frac{\bar{u}}{\bar{u}_\delta} + \frac{\bar{T}_\delta - \bar{T}_{\bar{r}}}{\bar{T}_\delta} \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2, \bar{T}_{\bar{r}} = \bar{T} + \bar{r} \frac{\bar{u}^2}{2C_p}, \quad (4.6)$$

which has the same form as Crocco-Busemann relation and Walz's equation except the different value of  $\bar{r}$ . For an adiabatic wall, Eq. (4.6) reduces to the widely accepted Walz's equation. This general relationship shows that the mean temperature has a quadratic dependence on the mean streamwise velocity at arbitrary wall-temperature conditions, and the effect of wall temperature is only to change the value of  $\bar{r}$ . Note that Eq. (4.6) is universal for CCF, CPF and CTBL. Fig.2 and Fig.3 compare Eq. (4.6) with DNS data of CCF, CPF and CTBL, which show a collapse between the current theory and the DNS data. In Fig. 2, Eq. (4.6) displays a much better performance than Walz's equation for non-adiabatic CTBL.

$\bar{r}$  in Eq. (4.6) requires exact wall information to be calculated, which is inconvenient in the engineering application. Therefore, we investigate the influence of wall-temperature

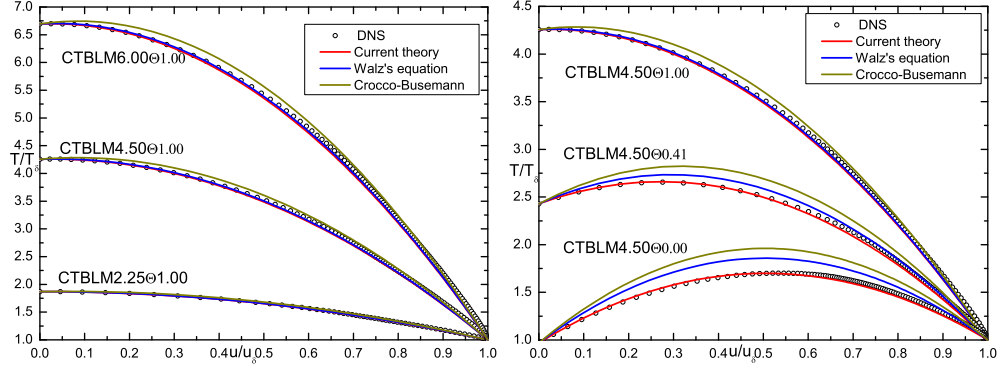


FIGURE 2. (color online). Mach number (left) and wall-temperature effects (right) of the temperature-velocity relationship Eq. (4.6) in CTBL.

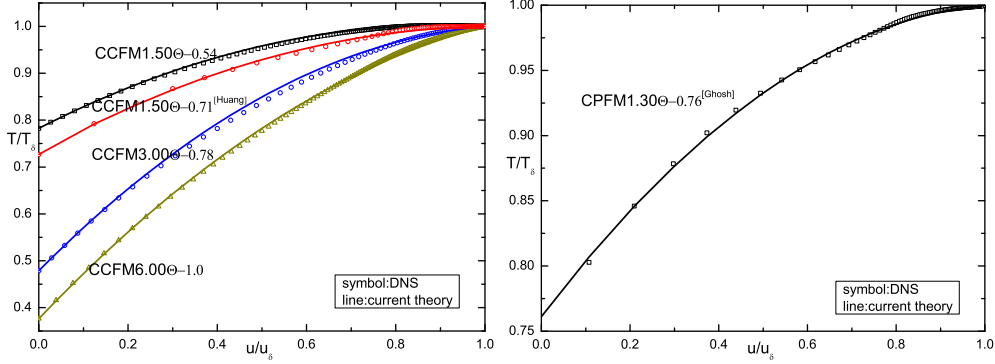


FIGURE 3. (color online). Comparison between the temperature-velocity relationship Eq. (4.6) and DNS data in compressible channel flows (left) and compressible pipe flows at different adiabatic degrees and Mach numbers. The DNS of  $CCFM1.50\Theta - 0.71$  comes from reference (Huang *et al.* 1995), and  $CPFM1.30\Theta - 0.76$  comes from reference (Ghosh *et al.* 2010).

on  $\bar{r}$ . To do this, we introduce the well-known Reynolds analogy factor defined as

$$s = \frac{2C_h}{C_f} = \frac{q_{yw}\bar{u}_\delta}{\bar{\tau}_w C_p (\bar{T}_w - \bar{T}_{aw})}, \quad (4.7)$$

where  $C_f = \frac{\bar{\tau}_w}{\bar{\rho}_\delta \bar{u}_\delta^2 / 2}$  is skin-friction coefficient,  $C_h = \frac{q_{yw}}{\bar{\rho}_\delta \bar{u}_\delta C_p (\bar{T}_w - \bar{T}_{aw})}$  is heat-transfer coefficient, i.e. Stanton number (Smits & Dussauge 2006). Using this definition, the general recovery factor  $\bar{r}$  can be rewritten as (see Appendix F)

$$\bar{r} = r[s \text{Pr} + (1 - s \text{Pr})\Theta]. \quad (4.8)$$

An outstanding property of Eq.(4.8) is that all the parameters in the right hand side of Eq. (4.8), except  $\Theta$ , are almost constant. In the literature (Bradshaw 1977; Duan *et al.* 2010; Duan & Martín 2011), the Reynolds analogy factor  $s$  varies between 1.0 and 1.2 without a clear trend with respect to  $\bar{T}_w$ ,  $Re$ ,  $M$ , chemical reaction and enthalpy condition. Here, we use  $s \text{Pr} = 0.81 \approx r^2$ , which is set by averaging the DNS data of us and several others (Duan *et al.* 2010; Duan & Martín 2011; Ghosh *et al.* 2010; Huang *et al.* 1995), as shown in Fig.(4). Using Eq. (4.8) and the definition of  $\Theta$ , Eq. (4.6) can be

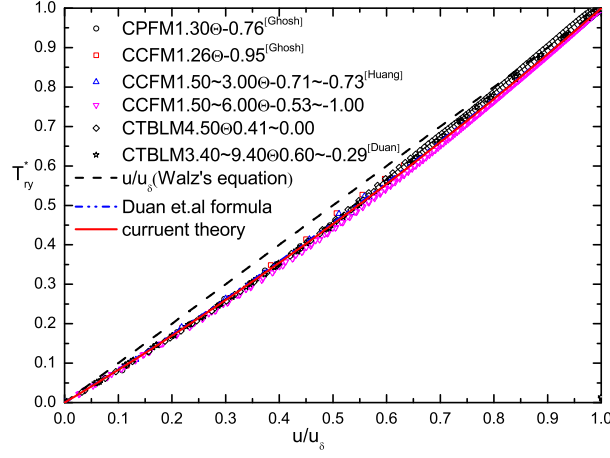


FIGURE 4. (color online). Variation of  $\bar{T}_{ry}^*$  (see Eq.(4.10)) versus  $\bar{u}/\bar{u}_\delta$  for CCF, CPF and CTBL. DNS of  $CTBLM3.40 \sim 9.00\Theta0.60 \sim -0.29$  comes from reference (Huang *et al.* 1995), and  $CCPFM1.30\Theta - 0.76$  comes from reference (Ghosh *et al.* 2010).

rearranged to (see Appendix G)

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_\delta} f\left(\frac{\bar{u}}{\bar{u}_\delta}\right) + \frac{\bar{T}_\delta - \bar{T}_r}{\bar{T}_\delta} \left(\frac{\bar{u}}{\bar{u}_\delta}\right)^2, f\left(\frac{\bar{u}}{\bar{u}_\delta}\right) = (1 - \alpha)\left(\frac{\bar{u}}{\bar{u}_\delta}\right)^2 + \alpha\left(\frac{\bar{u}}{\bar{u}_\delta}\right), \quad (4.9)$$

which is consistent with Duan et.al's empirical relation Eq. (2.29) except the slight difference between  $\alpha = sPr \approx 0.81$  in Eq.(4.9) and  $\alpha = 0.8259$  in Eq. (2.28)(see fig.(4)). Using Duan et.al (Duan & Martín 2011)'s (Duan & Martín 2011) definition about  $\bar{T}_{ry}^*$ , Eq. (2.29) can further be rearranged to:

$$\bar{T}_{ry}^* \equiv f\left(\frac{\bar{u}}{\bar{u}_\delta}\right) = (1 - \alpha)\left(\frac{\bar{u}}{\bar{u}_\delta}\right)^2 + \alpha\left(\frac{\bar{u}}{\bar{u}_\delta}\right), \alpha = sPr \approx r^2 \approx 0.81. \quad (4.10)$$

In this form, the  $\bar{T}_{ry}^*$  is a deterministic function of  $\bar{u}/\bar{u}_\delta$ . Hence, a collapse of different DNS data is expected. This is confirmed by fig.4, where the non-adiabatic DNS data of CPF, CCF and CTBL are plotted together. Indeed, an excellent collapse of DNS data to Eq.(4.10) is observed in the range of  $M = 1.26 \sim 9.4$  and  $\Theta = 0.6 \sim -1.0$ . Since the DNS by Duan et.al also considered the effect of real gas (enthalpy) and surface catalysis, such a collapse shows that the validity of Eq. (4.6) is independent on  $\bar{T}_w$ ,  $Re$ ,  $M$ , flow systems surface catalysis and enthalpy condition. Noting that Duan et al. (Duan & Martín 2011) already observed that the temperature-velocity relationship was influenced by both enthalpy condition and surface catalysis, here we clarify that this influence acts by changing the wall temperature (thus  $\bar{r}$ ), without involving any more complex reason. Additionally, we point out that the collapse in Fig.4 can only be realized by setting  $r \approx 0.9$  for CPF and CCF, which means that wall-bounded turbulence shares the same adiabatic recovery factor.

## 5. Fluctuation relationships based on the GRA

From Eq.(3.29) and Eq.(3.30), we have

$$T + \left( \frac{1}{\bar{u}} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \frac{(\overline{\rho v})' T'}{\bar{u}(\overline{\rho v})' u'} \right) \frac{u^2}{2} - \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w u - \varepsilon' \equiv \bar{T}_w \quad (5.1)$$

$$\bar{T} + \left( \frac{1}{\bar{u}} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \frac{(\rho v)'T'}{\bar{u}(\rho v)'u'} \right) \frac{\bar{u}^2}{2} - \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \bar{u} \equiv \bar{T}_w \quad (5.2)$$

Subtracting Eq.(5.2) from Eq.(5.1) and using the approximations in Eq.(2.4), we obtain

$$T' = \frac{(\rho v)'T'}{(\rho v)'u'} u' + \varepsilon' \quad (5.3)$$

If the contribution from  $\varepsilon'$  to the r.m.s. of  $T'$  is negligible, Eq. (5.3) yields

$$\frac{\sqrt{T'^2}}{\sqrt{u'^2}} \approx \pm \frac{(\rho v)'T'}{(\rho v)'u'} \approx \pm \frac{\overline{\rho v'T'}}{\overline{\rho v'u'}}, \quad (5.4)$$

where subtraction is for CTBL and plus for CPF/CCF, a convention in this subsection. The different signs are owing to the different temperature distribution along the wall-normal direction between CTBL and CPF/CCF. In CTBL, the high-temperature region is located near the wall, where the correlations  $\overline{\rho v'T'}$  and  $\overline{\rho v'u'}$  have opposite signs because of the ejections of low-speed/high-temperature fluids and the sweeps of high-speed/low-temperature fluids (Duan *et al.* 2010; Pirozzoli & Bernardini 2011). An opposite situation happens in CCF and CPF where the low-temperature region is located near the wall. As pointed out by Guarini *et al.* (Guarini *et al.* 2000), Eq.(5.4) expresses an analogy between the rates of turbulent heat and momentum transfer. If we further introduce the approximations of  $(\rho v)'T' \approx \bar{\rho} v'T'$  and  $(\rho v)'u' \approx \bar{\rho} v'u'$ , Eq. (5.4) yields  $R_{v'u'} \approx \pm R_{v'T'}$ , which is the basis of HSRA (Guarini *et al.* 2000). The equal correlations reveal that the turbulence-induced wall-normal transfer of heat is closely associated with that of momentum. In other words, the temperature field is a passive scalar field, consistent with Morkovin's viewpoint (Morkovin 1962).

Combining Eq. (2.4) and Eq. (5.4), a relationship between and can be obtained:

$$\frac{\sqrt{T'^2}/\bar{T}}{(\gamma - 1) Ma^2 \sqrt{u'^2}/\bar{u}} = \frac{1}{\pm \overline{Pr_t}(1 - \partial \bar{T}_t / \partial \bar{T})}. \quad (5.5)$$

This relation has the same form as that of HSRA, except replacing  $c = Pr_t$  in Eq. (1.2) with  $c = \overline{Pr_t}$ . Here, we call Eq.(5.5) the modified HSRA (MHSRA). A comparison between DNS and Eq.(5.5) is demonstrated for CTBL in Fig.5. To visualize clearly, Fig.5 plots the ratio of the left-hand side of SRA, HSRA and MHSRA to their right-hand side at different  $M$  and  $\Theta$ . Fig.5 shows that MHSRA is in excellent agreement with the DNS data throughout the boundary layer, while SRA is valid only to adiabatic wall and below  $0.7\delta$ . For SRA, an increased deviation from unity is observed in flows with colder walls. HSRA is plotted in the inset of the right figure. It is shown that MHSRA has a slight improvement over HSRA.

Using Eq.( 5.5), we can explain the validity of Eq.(2.18) at adiabatic wall, which is said to satisfy under an incorrect assumption as mentioned in sec.2.2. With Eq.( 4.5), a simple calculation gives

$$\frac{1}{\pm \overline{Pr_t}(\partial \bar{T}_t / \partial \bar{T} - 1)} \approx \frac{C_p}{\overline{Pr_t}} \frac{1}{\bar{u}} \frac{\partial \bar{T}}{\partial \bar{u}} = \frac{C_p}{\bar{u}} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w - \bar{r}. \quad (5.6)$$

For quasi-adiabatic wall, the right hand side of this relation is  $r \approx 0.9$ , close to the unity value of SRA, and, subsequently, leading to the good performance of SRA. For non-adiabatic wall, however, the deviation from unity, caused by local modification  $\frac{C_p}{\bar{u}} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w$



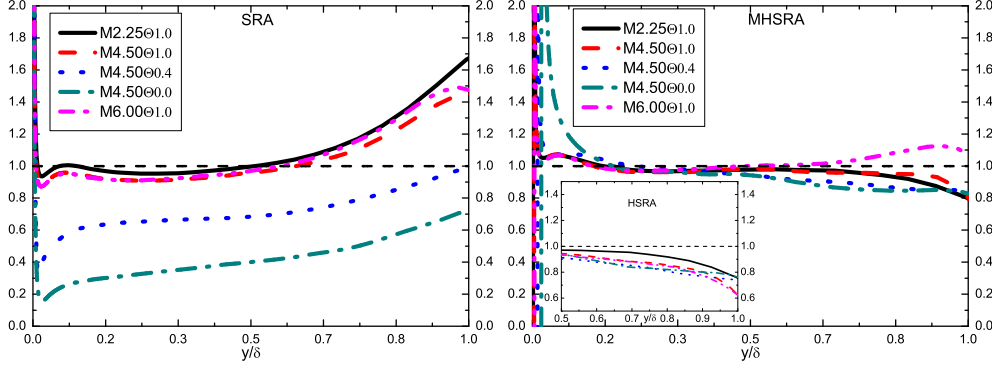


FIGURE 5. (color online). A comparison between SRA (left) and MHSRA (right) at different  $M$  and  $\Theta$ . The inset in the right figure is HSRA.

and global change of  $\bar{\tau}$ , becomes more and more significant with the increase of heat flux at wall, resulting in the failure of SRA.

For the correlation coefficient  $|R_{u'T'}|$ , on one hand, the existence of high-order perturbation  $\varepsilon'$  in the instantaneous fluctuation relation Eq. (5.3) excludes the possibility of  $|R_{u'T'}| = 1$ , an incorrect conclusion of SRA, ESRA, GSRA and HSRA; on the other hand, it prevents a simple formulation about  $|R_{u'T'}|$ . To avoid this dilemma, we recur to MHSRA. Simplifying the left-hand-side of Eq.(5.5), we have  $(\bar{u}/C_p)(\sqrt{\bar{u}'^2}/\sqrt{\bar{T}'^2}) = \pm \overline{\text{Pr}_t}(\partial \bar{T}_t / \partial \bar{T} - 1)$ . Substitution of this relation into Eq.(2.23) yields

$$R_{u'T'} = \pm \frac{1}{2} \left[ \frac{\bar{T}_t'^2 / \bar{T}'^2 - 1}{\overline{\text{Pr}_t}(\partial \bar{T}_t / \partial \bar{T} - 1)} - \overline{\text{Pr}_t}(\partial \bar{T}_t / \partial \bar{T} - 1) \right], \quad (5.7)$$

which gives a connection between  $R_{u'T'}$  and  $\overline{\text{Pr}_t}$ . Theoretically speaking, Eq.(5.7) has the same accuracy as that of MHSRA.

## 6. Discussion

Now we discuss the key approximations and assumptions involved in GRA. For CCF and CPF, GRA is a full analogy solution if the unique assumption of negligible  $\varepsilon'$  is valid. For CTBL, additional assumptions of negligible  $f_{res2} = 0$  and streamwise derivatives are applied to give an approximate analogy solution. The assumption about  $\varepsilon'$  is indirectly confirmed by the excellent agreement between DNS and GRA. The approximation of negligible streamwise derivatives is widely adopted in the study of turbulent boundary layers, which requires that the flows are slowly developing in the streamwise direction. The requirement prohibits an incautious application of GRA in CTBLs that have strong acceleration or temperature gradient along the wall, large local wall curvature, intense interactions with an impinging shock or a protruding object, etc. The self-consistence of approximation  $f_{res2} \approx 0$  can be verified. Under the solution of GRA, the residual error  $f_{res2}$  is equivalent to  $f_{res2} = \bar{\tau} \partial_y [(1 - \bar{r})\bar{u}] + (\text{Pr} - 1) \partial_y (\bar{q}_y)$  (see Appendix H). Approximation  $f_{res2} \approx 0$  is equivalent to require that  $f_{res2}$  is a small term compared with the dominant terms of Eq.(3.23), which is supported by DNS data (see Fig.6). Note  $f_{res2}$  is negligible above the near-wall region, where  $\bar{\tau} \partial_y \bar{u}$  and  $\partial_y (\bar{q}_y)$  is not important and  $\bar{r} \sim \text{Pr}$ . A possible deviation of GRA from DNS may observed in the near wall region where a small residual of  $f_{res2}$  exist (see Fig.6). This deviation increases for cold walls that have large  $\partial_y (\bar{q}_y)$  near the wall, as shown in Fig. 5. However, the difference between

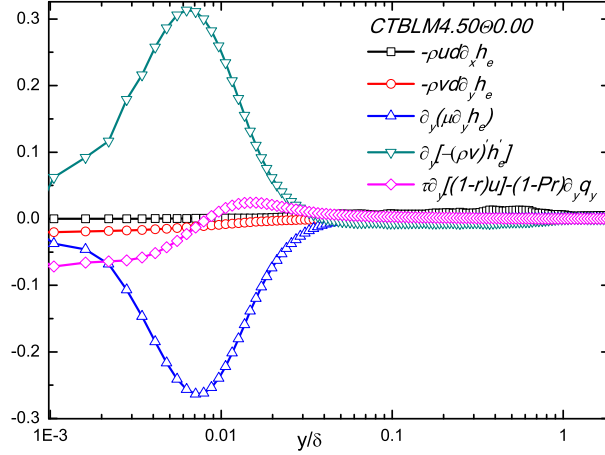


FIGURE 6. (color online). Distributions of the budgets of Eq.(3.23) along the wall-normal direction. The budgets are renormalized by the free-stream value.

MHSRA and DNS in the near wall region is not observed to the mean temperature-velocity relationship of GRA. Finally, for the mean relationship, the current theory is independent on  $Pr$  but requires  $\overline{Pr}_t = \text{const}$ , and the approximation of unity  $\overline{Pr}_t$  gives a rather accurate mean relationship as shown in Sec.4. It is worthy investigating whether or not the assumption  $\overline{Pr}_t = \text{const}$  works in other flow systems.

GRA demonstrates universality to various compressible wall-bounded turbulent flows under various wall temperature conditions. For the mean temperature-velocity relationship, GRA shows that the coupling between the mean velocity and mean thermal fields is same for various flows: the general total enthalpy (temperature)  $\bar{h}_g(\bar{T}_g)$  keeps the constant value of  $\bar{h}_{gw}(\bar{T}_w)$  along the wall-normal direction.  $\bar{h}_g(\bar{T}_g)$  successfully integrates the effects of viscous dissipation and heat flux through the wall on the energy transfer between kinetic energy and fluid enthalpy in wall-bounded flows. For the fluctuation relationship, the unique difference between internal flow (CCF/CPF) and external flow (CTBL) is the sign in Eq.(5.5). As discussed in sec.5, the difference is caused essentially by the different distributions of temperature, a passive scalar in nature determined by boundary conditions.

The universality of GRA needs a further discussion. We argue that the reason behind is the common turbulence mechanism in wall-bounded turbulence. Specifically, the essential dynamics of large scale eddy movements or coherent structures in wall-bounded turbulent flows are similar, without significantly influenced by compressibility, wall-temperature condition and flow situation far away from the wall. The turbulence transport of momentum and heat is dominated by the invariant vortical structures; thus a universal analogy solution for the velocity and temperature fields, formally like Walz's equation that is valid only to adiabatic CTBL and Duan's relation that applies to many more flows, is promised. GRA seems to be such a universal solution.

The adiabatic degree  $\Theta$  introduced in GRA is a well-founded dimensionless parameter for characterizing the wall-temperature effects in non-adiabatic compressible wall-bounded turbulent flows. In these flows, three control parameters are generally involved:  $M$ ,  $Re$ , and a dimensionless wall temperature that, we argue, has not been well defined up to now. To study the  $M$ -effect, which is an overwhelming topic in the literature of compressible turbulence, one has to compare different  $M$  flows at a same  $Re$  and a same dimensionless wall temperature, whose definitions are not trivial owing to the nonlinear

couplings between the different physical mechanisms they represent. We recently proposed a definition of  $Re_{\delta_{vw}}$  and showed it was appropriate for studying the  $M$  effects in adiabatic CTBLs. Note that the adiabatic degree  $\Theta$  in Eq.(4.8) is the unique parameter determining  $\bar{r}$ , and hence determining the temperature-velocity relationship of Eq. (4.6). So  $\Theta$  poses as a well-founded dimensionless parameter that can be applied to study the  $M$ -effects of non-adiabatic compressible wall-bounded turbulence.

The current theory can be applied to make quantitative predictions of the mean fields of compressible wall-bounded turbulence. A long lasting wish is to quantify the compressible wall-bounded turbulence with the much more fruitful results of incompressible wall-bounded turbulence, such as the recent formulations obtained by She et al. on incompressible channel flow, pipe flow and turbulent boundary layer (She *et al.* 2011a,b, 2010). The key question is to quantify the  $M$  and  $\Theta$  effects. For an adiabatic compressible wall-bounded turbulent flow (i.e.  $\Theta = 1$ ), the influence of  $M$  can be accounted by considering the variation of the mean fluid properties, which is the well-known Morkovin's hypothesis. Recently, we specified this hypothesis by considering both the mean density and viscosity and successfully removed the  $M$ -effects of a series of important flow quantities (You-sheng *et al.* 2012). Because of the nonlinear coupling between the velocity and thermal fields, these flow quantities cannot be predicted by those of the corresponding incompressible flows without the information of the mean fluid properties. GRA documented in this paper provides a solution for this problem. Now, combining GRA and the  $M$ -invariants rescaled by the mean thermal quantities, we are able to predict the mean profiles of velocity, temperature, density, as well as the Reynolds stress of compressible wall-bounded turbulence by using the formulations in the corresponding incompressible flows. We will report the results elsewhere.

## 7. Conclusions

In summary, we presented a general Reynolds analogy (GRA) between the Reynolds averaged momentum equation and energy equation of compressible wall-bounded turbulent flows. A full analogy is obtained for CPF and CCF, and an approximate analogy is obtained for CTBL. Several relations between the temperature and velocity fields are derived by GRA and validated by DNS data of CPF, CCF and CTBL. It is shown that the GRA is superior to the previous analogy theories for its higher accuracy, invariance with wall temperature condition and flow system, and solid physical basis. The GRA opens a door to the long lasting wish of predicting the mean fields of compressible wall-bounded turbulence with the information of the corresponding incompressible flow. The method to derive GRA may also be suitable for the study of other flow systems associated with thermal fields, such Rayleigh-Benard flow.

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## Appendix A.

Reference(Ghosh *et al.* 2010) gives Navier-Stokes equations in cylindrical coordinates, of which continue equation, streamwise momentum equation and energy equation can be averaged in time, axial and azimuthal direction to give, respectively,

$$\partial_y[\hat{y}\overline{\rho v}] = 0 \quad (\text{A } 1)$$

$$\partial_y[\hat{y}\overline{\rho uv}] = -\hat{y}\partial_x\bar{p} + \partial_y[\hat{y}\overline{\tau_{xy}}] \quad (\text{A } 2)$$

$$\partial_y[\hat{y}\overline{\rho h_t v}] = \partial_y[\hat{y}\overline{k\partial_y T}] + \overline{v\tau_{yy}} + \overline{u\tau_{xy}} \quad (\text{A } 3)$$

Continue equation means  $\overline{\rho v} = 0$ , and then one has  $\overline{\rho v f} = \overline{\rho v} \bar{f} + \overline{(\rho v)' f'} = \overline{(\rho v)' f'}$ . Substitution of this relation into momentum and energy equation gives immediately equation Eq.(3.1) and Eq.(3.2).

## Appendix B.

Using the definition about  $h_e$  and approximation Eq.(2.4), one has

$$\begin{aligned} \bar{\mu}\bar{u}\partial_y\bar{u} &\approx \bar{\mu}\partial_y[\bar{h}_e - (\bar{r} - 1)\bar{u}^2/2 - C_p\bar{T}] \\ &= \bar{\mu}\partial_y(\bar{h}_e) - (\bar{r} - 1)\bar{u}(\bar{\mu}\partial_y\bar{u}) - \bar{\mu}(\bar{u}^2/2)\partial_y\bar{r} - \text{Pr}\bar{k}\partial_y\bar{T} \end{aligned} \quad (\text{B } 1)$$

$$\overline{(\rho v)' h'_t} = \overline{(\rho v)' (h_e - (\bar{r} - 1)u^2/2)'} \approx \overline{(\rho v)' h'_e} - (\bar{r} - 1)\bar{u}\overline{(\rho v)' u'} \quad (\text{B } 2)$$

Substituting the two relation into Eq.(3.2) gives

$$0 \approx \partial_y[\hat{y}(\bar{\mu}\partial_y(\bar{h}_e) - (\bar{r} - 1)\bar{u}\bar{\tau} - \bar{\mu}(\bar{u}^2/2)\partial_y\bar{r} - \text{Pr}\bar{k}\partial_y\bar{T} - \overline{(\rho v)' h'_e} + \overline{k\partial_y T})] \quad (\text{B } 3)$$

Rearrangement last relation gives Eq.(3.6)

## Appendix C.

Integration the constraint Eq.(3.8) from center to  $y$  gives

$$\hat{y}[(\bar{r} - 1)\bar{u}\bar{\tau} + (\bar{\mu}\bar{u}^2/2)\partial_y\bar{r} + (1 - \text{Pr})\bar{q}_y - U_w\bar{\tau}] \approx \text{const} \quad (\text{C } 1)$$

In the center, the symmetry imposes  $\bar{q}_y\delta = 0$ ,  $(\partial_y\bar{r})\delta = 0$  and  $\bar{\tau}\delta = 0$ , and naturally  $\text{const} = 0$ . Then Eq.(C 1) becomes

$$(\bar{r} - 1)\bar{u}\bar{\tau} + (\bar{\mu}\bar{u}^2/2)\partial_y\bar{r} + (1 - \text{Pr})\bar{q}_y - U_w\bar{\tau} \approx 0 \quad (\text{C } 2)$$

After multiplying Eq.(3.14) by  $\bar{\mu}$ , one has

$$(\bar{\mu}\bar{u}^2/2)\partial_y\bar{r} = (U_w - \bar{r}\bar{u})\bar{\mu}\partial_y\bar{u} + \text{Pr}\bar{q}_y \quad (\text{C } 3)$$

Substituting this relation and Eq.(3.7) into Eq.(3.8) yields

$$(\bar{r} - 1)[C_p\overline{(\rho v)' T'} + \bar{q}_y] + (U_w - \bar{r}\bar{u})\bar{\mu}\partial_y\bar{u} + \text{Pr}\bar{q}_y + (1 - \text{Pr})\bar{q}_y - U_w\bar{\tau} \approx 0 \quad (\text{C } 4)$$

Rearrangement of last equation with the definition about  $\bar{\tau}$  gives

$$(\bar{r} - 1)[C_p\overline{(\rho v)' T'}] + \bar{r}(\bar{q}_y - \bar{u}\bar{\tau}) + (U_w - \bar{r}\bar{u})\overline{(\rho v)' u'} \approx 0 \quad (\text{C } 5)$$

With Eq.(3.7), one has  $\bar{q}_y - \bar{u}\bar{\tau} = -C_p\overline{(\rho v)' T'}$ , substitution of which into last equation gives

$$C_p\overline{(\rho v)' T'} - (U_w - \bar{r}\bar{u})\overline{(\rho v)' u'} \approx 0. \quad (\text{C } 6)$$

Solving this equation in terms of  $\bar{r}$  gives Eq.(3.16)

## Appendix D.

Using the definition about  $h_e$ , approximation Eq.(2.4), Eq.(B 1) and Eq.(B 2), Eq.(3.22) becomes

$$\begin{aligned} \bar{\rho}\bar{u}\partial_x[\bar{h}_e - (\bar{r} - 1)\bar{u}^2/2] + \overline{\rho v}\partial_y[\bar{h}_e - (\bar{r} - 1)\bar{u}^2/2] &\approx \\ \partial_y[\bar{\mu}\partial_y(\bar{h}_e) - (\bar{r} - 1)\bar{u}\bar{\tau} - \bar{\mu}(\bar{u}^2/2)\partial_y\bar{r} - \text{Pr}\bar{k}\partial_y\bar{T} - \overline{(\rho v)' h'_e}] &+ \partial_y(\bar{k}\partial_y\bar{T}) \end{aligned} \quad (\text{D } 1)$$

Rearrangement of last equation gives

$$\overline{\rho u} \partial_x \bar{h}_e + \overline{\rho v} \partial_y \bar{h}_e \approx \partial_y [\bar{\mu} \partial_y (\bar{h}_e) - \overline{(\rho v)' h'_e}] + f_{res}, \quad (\text{D } 2)$$

where

$$\begin{aligned} f_{res} = & (\bar{r} - 1) \bar{u} \overline{\rho u} \partial_x \bar{u} + (\bar{r} - 1) \overline{\rho v} \partial_y \bar{u} + \overline{\rho v} \bar{u}^2 \partial_y \bar{r} / 2 \\ & + \partial_y [(\text{Pr} - 1) \bar{q}_y - \bar{u}^2 \bar{\mu} \partial_y \bar{r} / 2 + (1 - \bar{r}) \bar{u} \bar{\tau}]. \end{aligned} \quad (\text{D } 3)$$

Note that the streamwise derivative  $\overline{\rho u} \partial_x \bar{u}$  can be written with normal derivative by momentum equation, i.e.  $\overline{\rho u} \partial_x \bar{u} \approx \partial_y \bar{\tau} - \overline{\rho v} \partial_y \bar{u}$ . Then can be further written as

$$\begin{aligned} f_{res} = & [(\bar{r} - 1) \bar{u}] [\partial_y \bar{\tau} - \overline{\rho v} \partial_y \bar{u}] + [(\bar{r} - 1) \bar{u}] \overline{\rho v} \partial_y \bar{u} + [\overline{\rho v} \bar{u}^2 / 2] \partial_y \bar{r} \\ & + \partial_y [(\text{Pr} - 1) \bar{q}_y - \bar{\mu} (\bar{u}^2 / 2) \partial_y \bar{r} + (1 - \bar{r}) \bar{u} \bar{\tau}]. \end{aligned} \quad (\text{D } 4)$$

The cancellation of  $[(\bar{r} - 1) \bar{u}] \overline{\rho v} \partial_y \bar{u}$  in last equation gives Eq.(3.24).

## Appendix E.

Rearrangement of Eq.(3.24) gives

$$f_{res} = (\overline{\rho v} / \bar{\mu}) (\bar{u}^2 \bar{\mu} \partial_y \bar{r} / 2) + \partial_y [(\text{Pr} - 1) \bar{q}_y - (\bar{u}^2 \bar{\mu} \partial_y \bar{r} / 2)] + \bar{\tau} \partial_y [(1 - \bar{r}) \bar{u}]. \quad (\text{E } 1)$$

After replacing  $\bar{u}^2 \bar{\mu} \partial_y \bar{r} / 2$  with Eq.(C 3), a further rearrangement yields

$$\begin{aligned} f_{res} = & (\overline{\rho v} / \bar{\mu}) [(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y] + \partial_y [\bar{u} \bar{\tau} - \bar{q}_y - (U_w - \bar{r} \bar{u}) \overline{(\rho v)' u'}] \\ & + [(\bar{r} - 1) \bar{u} - U_w] \partial_y \bar{\tau}. \end{aligned} \quad (\text{E } 2)$$

Therefore, one has

$$\begin{aligned} f_{res} = & \partial_y [C_p \overline{(\rho v)' T'} - (U_w - \bar{r} \bar{u}) \overline{(\rho v)' u'}] + [(\bar{r} - 1) \bar{u} - U_w] \partial_y \bar{\tau} \\ & + \partial_y [\bar{u} \bar{\tau} - C_p \overline{(\rho v)' T'} - \bar{q}_y] + (\overline{\rho v} / \bar{\mu}) [(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y]. \end{aligned} \quad (\text{E } 3)$$

A decomposition gives  $f_{res1}$  and  $f_{res2}$

## Appendix F.

With Eq.(4.5), the  $\bar{r}$  can be further written as

$$\begin{aligned} \bar{r}(y) = & \frac{\bar{T}_r - \bar{T}_\delta}{\bar{u}_\delta^2 / (2C_p)} \left( \frac{\bar{T}_w - \bar{T}_\delta}{\bar{T}_r - \bar{T}_\delta} + \frac{\bar{u}_\delta}{\bar{T}_r - \bar{T}_\delta} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) \\ = & r \left( \frac{\bar{T}_w - \bar{T}_\delta}{\bar{T}_r - \bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_r - \bar{T}_\delta} \frac{\bar{u}_\delta}{\bar{T}_r - \bar{T}_w} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w \right) \end{aligned} \quad (\text{F } 1)$$

Because of  $\frac{\bar{u}_\delta}{\bar{T}_r - \bar{T}_w} \frac{\partial \bar{T}}{\partial \bar{u}} \Big|_w = sPr$ ,  $\frac{\bar{T}_r - \bar{T}_w}{\bar{T}_r - \bar{T}_\delta} = 1 - \frac{\bar{T}_w - \bar{T}_\delta}{\bar{T}_r - \bar{T}_\delta}$  and  $\frac{\bar{T}_w - \bar{T}_\delta}{\bar{T}_r - \bar{T}_\delta} = \Theta$ , one has

$$\bar{r}(y) = r[\Theta + (1 - \Theta)sPr] = r[sPr + (1 - sPr)\Theta] \quad (\text{F } 2)$$

## Appendix G.

With the definition about  $\bar{T}_{\bar{r}}$  and  $\bar{T}_r$ , one has

$$\begin{aligned} \bar{T}_{\bar{r}} = & \bar{T}_r + (\bar{r}/r - 1)(r\bar{u}_\delta^2 / (2C_p)) = \bar{T}_r + (sPr - 1)(1 - \Theta)(\bar{T}_r - \bar{T}_\delta) \\ = & \bar{T}_r + (sPr - 1)(\bar{T}_r - \bar{T}_w) \end{aligned} \quad (\text{G } 1)$$

Substituting this relation into Eq.(4.6), one has

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w + (s \text{Pr} - 1)(\bar{T}_r - \bar{T}_w)}{\bar{T}_\delta} \frac{\bar{u}}{\bar{u}_\delta} + \frac{\bar{T}_\delta - \bar{T}_r - (s \text{Pr} - 1)(\bar{T}_r - \bar{T}_w)}{\bar{T}_\delta} \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2 \quad (\text{G } 2)$$

Rearrangement of last equation gives

$$\frac{\bar{T}}{\bar{T}_\delta} = \frac{\bar{T}_w}{\bar{T}_\delta} + \frac{\bar{T}_r - \bar{T}_w}{\bar{T}_\delta} \left[ s \text{Pr} \frac{\bar{u}}{\bar{u}_\delta} + (1 - s \text{Pr}) \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2 \right] + \frac{\bar{T}_\delta - \bar{T}_r}{\bar{T}_\delta} \left( \frac{\bar{u}}{\bar{u}_\delta} \right)^2 \quad (\text{G } 3)$$

,which gives Eq.(4.9)

## Appendix H.

Using the relation Eq.(3.27), Eq.(3.26) can be rewritten as

$$f_{res2} = \partial_y [\bar{u} \bar{\tau} - (U_w - \bar{r} \bar{u})(\bar{\mu} \partial_y \bar{u} - \bar{\tau}) - \bar{q}_y] + [(\bar{r} - 1) \bar{u} - U_w] \partial_y \bar{\tau} + (\bar{\rho} \bar{v} / \bar{\mu}) [(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y] \quad (\text{H } 1)$$

Rearrangement of this equation gives

$$f_{res2} = \bar{\tau} \partial_y [(1 - \bar{r}) \bar{u}] + (\text{Pr} - 1) \partial_y (\bar{q}_y) + \partial_y [(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y] + (\bar{\rho} \bar{v} / \bar{\mu}) [(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y] \quad (\text{H } 2)$$

With the fourth term of Eq.(4.5), one has  $(U_w - \bar{r} \bar{u}) \bar{\mu} \partial_y \bar{u} + \text{Pr} \bar{q}_y = 0$ . Then  $f_{res2}$  can be further simplified to

$$f_{res2} = \bar{\tau} \partial_y [(1 - \bar{r}) \bar{u}] + (\text{Pr} - 1) \partial_y (\bar{q}_y) \quad (\text{H } 3)$$

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